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REMARKS ON THE DOCTRINE OF LIMITS.

BY PROFESSOR SIMON NEWCOMB.

THE very able papers on the doctrine of limits by Professors Judson and Wood in the ANALYST for July 1881 and May 1882 are suggestive of the necessity of a more complete examination of the subject if all possible misapprehension is to be avoided. In the use of any system of mathematical nomenclature a great number of unexpressed hypotheses have to be understood as limiting the statements, and the difficulty is to have these hypotheses equally understood by the writer and the reader. There is a kind of mathematical common sense which has to be taken as the basis of such a discussion, but it is very difficult to see just how far we are to assume that this sense will lead us without reasoning

In the paper referred to it appears to me that the propositions laid down by the respective writers are all correct when correctly interpreted but that their criticisms upon others arise in part from misapprehension. Professor Judson's first five propositions appear not open to any criticism and the following three to be only a definition of what shall be understood by a variable decreasing without limit.

But when, page 109, he shows that Whewell's axiom, *Whatever is true up to the limit is true at the limit,* is unsound, he makes too sweeping a use of the word *whatever.*

The axiom is obviously intended to apply only to finite quantitative relat'ns and therefore does not apply to any of the cases which he assumes. These case are descriptive, not quantitative.

Professor Wood's ingenious *reductio ad absurdum* of my reasoning on the sum of an infinite geometrical progres'n admirably illustrates what I wish to say on the unexpressed hypotheses which lie at the basis of the propositions of limits. He takes my reasoning that a debt can never be wholly disch'd by continually paying one half of it, changes the reading of it so as to make it apply to a prob. of his own in which a point moves a distance one half in half a minute, a distance one fourth in one fourth of a minute, and so on, and puts this changed reading into quotation marks in such a way as wo'ld lead the reader to suppose that I had reasoned in this way about this very problem*. No doubt Professor Wood considers the two problems perfectly analogous and to show their difference we must take up the unexpressed hypotheses. Let us first consider some definitions of the term limit. I take the following two.

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*The writer learns that the change of words alluded to was accidental. But the fact remains that Prof. Wood's problem is one that was never considered by the writer.

WOOD. "The limit of a variable is a quantity which the variable approaches and from which it may be made to differ by less than any assignable quantity." (ANALYST, p. 81.) This definition is too sweeping since it includes every possible value which the variable may take. If we suppose an angle to increase from 0° to 90° and if we assume any quantity k not greater than 1, the sine will in the course of the increase approach this quantity and differ from it by less than any assignable quantity. Hence k is a limit of the sine.

WENTWORTH. "When it can be shown that the value of a variable, meas'd at a series of definite intervals, can by indefinite continuation of the series be brought to differ from a given constant by less than any assigned quantity, without ever exceeding the constant, that constant is called the *Limit* of the variable, and the variable is said to *approach indefinitely to its limit.*" (*Geom.* p. 87.)

It seems to me that the word "exceeding" in this def. should be replaced by *equalling*, because the values of the variable may be alter'ly greater and less than the limit. Otherwise the def. has the adv. of being stated in such a manner as to indicate at once one of the hyp's commonly assumed in silence.

Now a variable quantity may in general have any value whatever. What then do we mean by speaking of a quantity which the variable can never reach or never exceed? Manifestly we must think of it as determined under certain conditions or subject to certain limitations, and the use of the word *limit* must relate to these conditions or relations.

The great object of the method of limits is *to determine quantities in cases where the ordinary operations with finite quantities fail in consequence of some of the quantities of the problem becoming infiinite in number, or of zero or infinity entering in such a way that the operations shall not be rigorously practicable.* Thus, the sum of an infinite series would require an infinity of additions and the calculation of the area of a circle by increasing the number of sides of a polygon would also require an infinity of operations. But if we transform the problem so as to avoid an infinity of operations it is a perversion of speech to apply the word limit to the result. But this is just what Professor Wood has done. In the case he supposes, of a point mov'g half the distance in half a minute, half the remaining distance in one fourth of a minute and so on, that is, of moving with a uniform motion, he entirely removes the problem from the class to which the method of limits applies. Of course the ambiguity of the word *never* comes in as a source of perturbation. Define the sense in which it is used, and the *reductio ad absurdum* becomes applicable only to Prof. Wood's way of considering the problem. When used in the method of limits it is understood to imply an indef. continuat'n of some series of operations which cannot be executed in any finite time, and of which we therefore say they can *never* be completely executed.

Answer to Query. (See p. 64 or 94.) *By Prof. C. A. Van Velzer, University of Wisconsin, Madison, Wis.*—In the determinant

$$\begin{vmatrix} 0 & b & c \\ b & 1 & a \\ c & a & 1 \end{vmatrix}$$

sub't from the third column $a \div b$ times the first column, and there results

$$\begin{vmatrix} 0 & b & c \\ b & 1 & 0 \\ c & a & 1 - ac \div b \end{vmatrix}.$$

In this determinant add to the first column $2ab$ times the third and we have

$$\begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c + 2ab - 2a^2c & a & 1 - ac \div b \end{vmatrix}.$$

In this determinant add to the third row $a \div b$ times the first, and subtract $2a$ times the second and we get

$$\begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix};$$

the required determinant, where, throughout, a is used for short in place of $\cos A$.

The transformation from this last determinant back to the first is also easy to effect.

Any determinant may in general be transformed into any other equivalent determinant of the same order. Let

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix}$$

and let us represent the first of these by $\triangle_a$ and the second by $\triangle_x$.

To the fourth column of $\triangle_a$ add l_1 times the first column, m_1 times the second and n_1 times the third, where l_1, m_1, n_1 are chosen to satisfy the eq's

$$\begin{aligned} l_1a_1 + m_1b_1 + n_1c_1 &= w_1 - d_1, \\ l_1a_2 + m_1b_2 + n_1c_2 &= w_2 - d_2, \\ l_1a_3 + m_1b_3 + n_1c_3 &= w_3 - d_3. \end{aligned}$$

By this means $\triangle_a$ is changed into

$$\begin{vmatrix} a_1 & b_1 & c_1 & w_1 \\ a_2 & b_2 & c_2 & w_2 \\ a_3 & b_3 & c_3 & w_3 \\ a_4 & b_4 & c_4 & \delta_4 \end{vmatrix}.$$

In this determinant add to the third column l_2 times the first, m_2 times the second and n_2 times the fourth, where l_2, m_2 and n_2 are chosen to satisfy the three equations

$$\begin{aligned} l_2a_1 + m_2b_1 + n_2w_1 &= z_1 - c_1, \\ l_2a_2 + m_2b_2 + n_2w_2 &= z_2 - c_2, \\ l_2a_3 + m_2b_3 + n_2w_3 &= z_3 - c_3, \end{aligned}$$

By this means we obtain the determinant

$$\begin{vmatrix} a_1 & b_1 & z_1 & w_1 \\ a_2 & b_2 & z_2 & w_2 \\ a_3 & b_3 & z_3 & w_3 \\ a_4 & b_4 & \gamma_3 & \delta_4 \end{vmatrix}.$$

In this determinant add to the second column l_3 times the first, m_3 times the third and n_3 times the fourth, where l_3, m_3, n_3 satisfy the equations

$$\begin{aligned} l_3a_1 + m_3z_1 + n_3w_1 &= y_1 - b_1, \\ l_3a_2 + m_3z_2 + n_3w_2 &= y_2 - b_2, \\ l_3a_3 + m_3z_3 + n_3w_3 &= y_3 - b_3. \end{aligned}$$

We thus arrive at the determinant

$$\begin{vmatrix} a_1 & y_1 & z_1 & w_1 \\ a_2 & y_2 & z_2 & w_2 \\ a_3 & y_3 & z_3 & w_3 \\ a_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix}.$$

Now add to the first column l_4 times the second, m_4 times the third and n_4 times the fourth where l_4, m_4 and n_4 satisfy the equations

$$\begin{aligned} l_4y_1 + m_4z_1 + n_4w_1 &= x_1 - a_1, \\ l_4y_2 + m_4z_2 + n_4w_2 &= x_2 - a_2, \\ l_4y_3 + m_4z_3 + n_4w_3 &= x_3 - a_3. \end{aligned}$$

This gives the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ a_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix}.$$

To the fourth row of this determinant add l_5 times the first, m_5 times the second, and n_5 times the third row, where l_5, m_5, n_5 satisfy the equations

$$\begin{aligned} l_5x_1 + m_5x_2 + n_5x_3 &= x_4 - a_4, \\ l_5y_1 + m_5y_2 + n_5y_3 &= y_4 - \beta_4, \\ l_5z_1 + m_5z_2 + n_5z_3 &= z_4 - \gamma_4. \end{aligned}$$

By this means the elements a_4, β_4, γ_4 are changed into x_4, y_4, z_4, and since the resulting determinant must equal $\triangle_x$, it follows that the element δ_4 will be changed into w_4 and the determinant $\triangle_x$ is thus arrived at.

The process here employed is evidently applicable to determinants of any order. The process is a definite one and leads to definite results unless some of the quantities l, m, n become infinite through the vanishing of the determinant of their coefficients, which, in every case, is a first minor of $\triangle_a$ or $\triangle_x$ or one of the intermediate determinants. Should this thing happen we must transform $\triangle_a$ not directly into $\triangle_x$ but into a determinant prepared from $\triangle_x$ so that the difficulty will be avoided. One case of difficulty, however, cannot be avoided by changing $\triangle_x$ beforehand and that is where

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

In this case, before we apply the general method we must change $\triangle_a$ so that the first minor corresponding to d_4 shall not vanish.

I believe that, in every case where the difficulty here spoken of arises, $\triangle_a$ or $\triangle_x$ or both may be changed, before applying the process of this paper, into determinants say $\triangle'_a$ and $\triangle'_x$ so that $\triangle'_a$ may be transformed into $\triangle'_x$ and therefore $\triangle_a$ into $\triangle_x$ If so, the transformation can *always* be effected; if not, it can *generally* be effected.

I have not yet found time to consider this last question.

[Professor L. G. Barbour answers Mr. Eastwood's Query (see p. 96) as follows:]

* * * "For much the same reason that it is allowable and sufficient to write $\log(1+y) = My+Ny^2+Py^3$ &c. It would not do to write $\log y = My+Ny^2+Py^3$ &c., nor $\log(1+y) = Ny^2+Py^3$ &c. And it it would not do to write $dp\div dt = q^2$, $dq \div dx = p^2$. At least there seems to be no antecedent likelihood that the former of these diff. coefficients could have been obtained from a combination of p, q and t. Nor could we write

$$\frac{dp}{dt} = pq, \frac{dq}{dx} = pq, \text{ for then } \frac{dp}{dt} \text{ must} = \frac{dq}{dx},$$

which is not a condition given. We then *assume* (if this term be prefer'd)

$$\frac{dp}{dt} = p^2, \frac{dq}{dx} = q^2,$$

which certainly fulfils the most obvious condition, viz.; that

$$\frac{dp}{dt}\cdot\frac{dq}{dx} = p^2q^2,$$

and does not present any difficulty on the threshhold. The integration procedes easily, and without encountering any unforeseen obstacle, to the complete primitives

$$e^{\psi} = \frac{c'}{t+c}; \quad e^{\psi} = \frac{c'''}{x+c''};$$

$$\therefore x = tc_4+c_5.$$

In the part of my solution omitted in printing, it was proven that there could be no singular solution. Hence it seems to the writer that the guarded phrase 'allowable and sufficient', instead of the customary 'necessary and sufficient', is sustained by analogy and confirmed by the result."